$x_1, x_2, \ldots, x_{10}$

average $\bar{x} = \dfrac{x_1 + x_2 + \cdots + x_{10}}{10}$

$\boxed{\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}}$

Section 6.5: Average Value of a Function

Objective: In this lesson, you learn

☐ How to define the average value of a continuous function on an interval and to establish the Mean Value Theorem for Integrals.

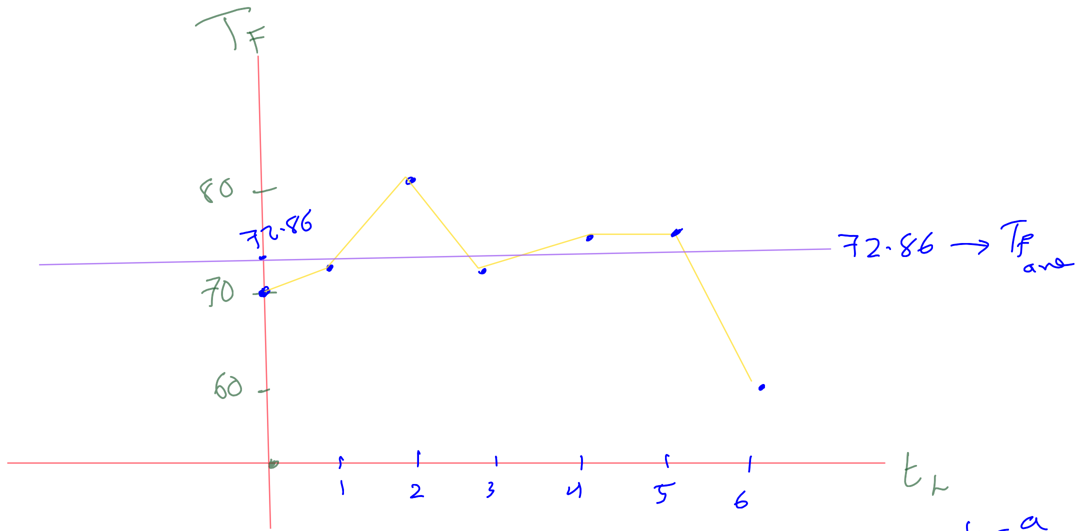
I. Average Value of a Function

It is easy to calculate the average of finitely many numbers, but how about the average of numbers in a set containing infinitely many elements?

Example 1: Calculate the average temperature, t-time in hours and T-temperature in Fahrenheit?

t_h	0	1	2	3	4	5	6
T_F	70	72	80	72	75	76	60

$$\text{Average } T_{F_{ave}} = \frac{70+72+80+72+75+76+60}{7} \simeq 72.86$$



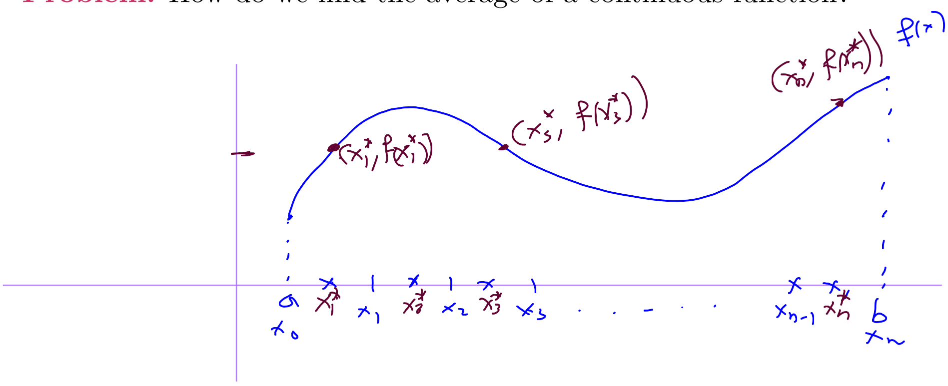
$n = \dfrac{b-a}{\Delta x}$

Problem: How do we find the average of a continuous function?

$\Delta x = \dfrac{b-a}{n}$

$[a,b] \xrightarrow{\text{into}}$ n-subintervals.

$[x_0, x_1], [x_1, x_2], \ldots [x_{n-1}, x_n]$



$$\text{Average of } f = f_{ave} \simeq \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \sum_{i=1}^{n} \frac{f(x_i)}{n}$$

$$f_{ave} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(x_i^*)}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(x_i^*)}{\frac{b-a}{\Delta x}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(x_i^*)\,\Delta x}{b-a}$$

$= \lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

If we want to find the average of a function $y = f(x)$, $a \le x \le b$:

1. First, divide the interval $[a, b]$ into n sub-intervals of equal length $\triangle x = \dfrac{b-a}{n}$.

2. Then choose samples points $x_1^*, \cdots, x_n^*$ in each sub-interval and calculate the average of the numbers

$$f(x_1^*) + \cdots + f(x_n^*)$$

as

$$[f(x_1^*) + \cdots + f(x_n^*)]/n.$$

Since $\Delta x = \frac{b-a}{n}$, $n = (b-a)/\Delta x$, so the average value becomes

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a}[f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

3. As n increases, we are computing the average value of a large number of closely spaced values. Thus, the limiting value is

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x,$$

the limit of a Riemann sum, so it is equal to

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Therefore, we define **the average value** of f on the interval $[a, b]$ as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 2: Find the average value of $f(x) = \sin x$ for $0 \le x \le 2\pi$?

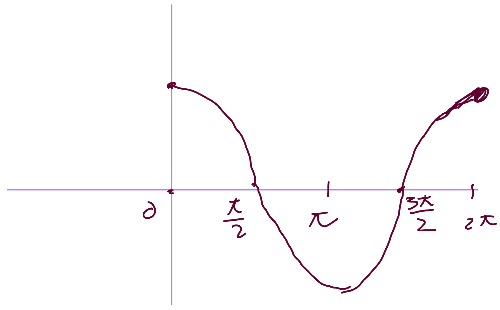
$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

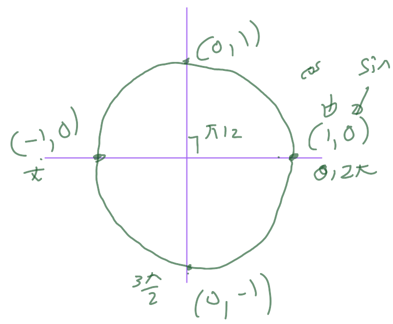
$$= \frac{1}{2\pi - 0}\int_0^{2\pi} \sin x \,dx$$

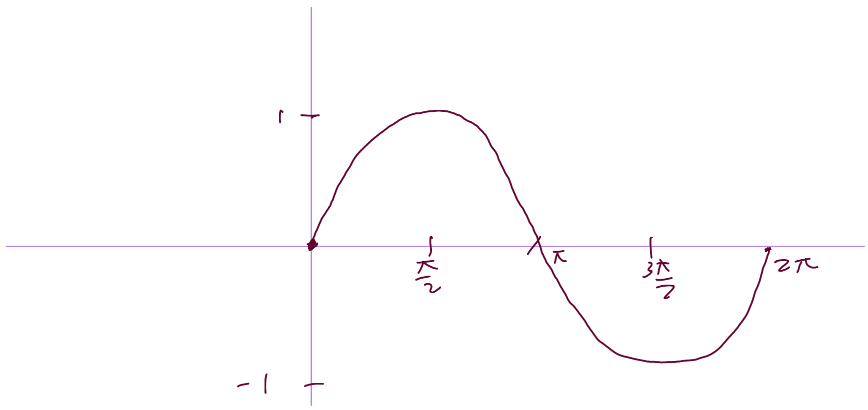
$$= \frac{1}{2\pi}\left(-\cos x \Big|_0^{2\pi}\right.$$

$$= \frac{-1}{2\pi}\left(\cos 2\pi - \cos 0\right)$$

$$= \frac{-1}{2\pi}\left(1 - 1\right) = 0$$





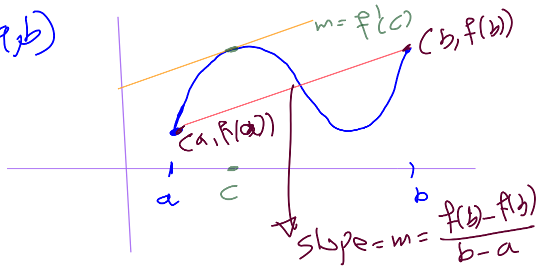


$f_{ave} = 0$

MVT

1. If f conts. on $[a,b]$, f is diff on (a,b) at least there is $c \in (a,b)$ s.t

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



slope $= m = \frac{f(b)-f(a)}{b-a} = f'(c)$

II. The Mean Value Theorem for Integrals

The next question is whether there is a specific number c in $[a,b]$ at which, $f_{ave} = f(c)$. The following theorem illustrates that the answer to the above question is affirmative for a continuous function.

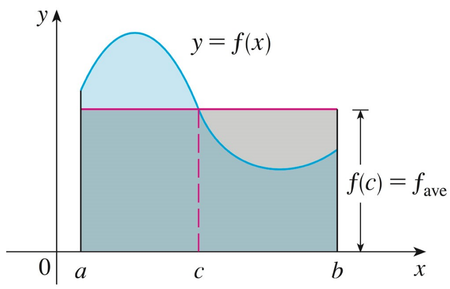
Theorem

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

that is, $\int_a^b f(x)\,dx = f(c)(b-a)$.

The geometric interpretation of the Mean Value Theorem for Integrals is that for positive functions f, there is a number c such that the rectangle with base $[a,b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.

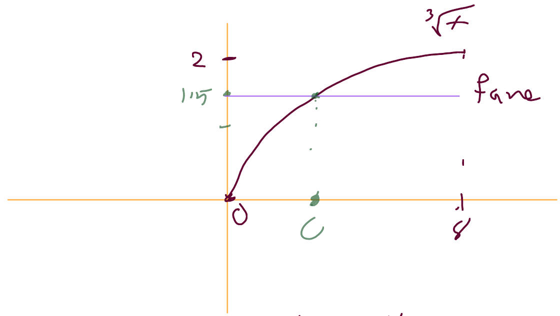


Example 3: Consider the function $f(x) = \sqrt[3]{x}$ on $[0,8]$

A) Find the average value of f on the given interval?

B) Fnd c such that $f_{ave} = f(c)$?

C) Sketch the graph of f and a rectangle whose area is the same as the area under the graph f?



A) $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$= \frac{1}{8-0}\int_0^8 \sqrt[3]{x}\,dx$

$= \frac{1}{8}\int_0^8 x^{1/3}\,dx = \frac{1}{8}\frac{3}{4}x^{4/3}\Big|_0^8 = \frac{3}{32}\cdot\left(8^{4/3} - 0^{4/3}\right)$

$= \frac{3}{32}(16) = \frac{3}{2} = 1.5$

$f_{ave} = 1.5$

$f(x) = \sqrt[3]{x}$

(2) $f(c) = f_{ave}$

$\left(\sqrt[3]{c}\right)^3 = (1.5)^3$

$c = (1.5)^3 \cong 3.375$

Check!: $f(c) = f\left(\left(1.5\right)^3\right) = \cancel{\sqrt[3]{(1.5)^3}} = 1.5 = f_{ave}.$

(3)